

# Canonical quantization of a massive Weyl field

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(ΩDated: July 5, 2011)

We construct a consistent theory of a quantum massive Weyl field. We start with the classical field theory description of Weyl spinors in frames of the Hamilton formalism. Then we carry out a canonical quantization of the system. It is found that a Weyl field can be quantized in two independent ways. We also propose the new interpretation of a quantized Weyl field. In particular the quantum analog of the classical Majorana condition is suggested.

PACS numbers: 03.65.Pm, 11.10.Ef, 14.60.Pq

Keywords: Weyl field, Hamilton formalism, quantization, Majorana neutrino

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Majorana particles are known to play an important role in the modern theoretical physics, especially in the studies of neutrinos. The most natural mechanism for the neutrino mass generation requires that neutrinos are Majorana particles [1]. Although presently there is no universally recognized experimental results casting light upon the nature of neutrinos, numerous attempts are made to investigate whether neutrinos are Dirac or Majorana particles [2].

It is well known that instead of dealing with a four component spinor  $\psi$  satisfying the Majorana condition,

$$\psi^c = \varkappa_c i \gamma^2 \psi^* = \psi, \quad (1)$$

the dynamics of a Majorana particle can be re-formulated in terms of the two component Weyl spinors. Here  $\varkappa_c$  is a phase factor having the unit absolute value. In our analysis we shall suppose that  $\varkappa_c = 1$ . The wave equations for the Weyl spinors have the form,

$$\dot{\eta} - c(\boldsymbol{\sigma} \nabla) \eta + \frac{mc^2}{\hbar} \sigma_2 \eta^* = 0, \quad (2)$$

or

$$\dot{\xi} + c(\boldsymbol{\sigma} \nabla) \xi - \frac{mc^2}{\hbar} \sigma_2 \xi^* = 0, \quad (3)$$

where  $m$  is the mass of the particle and  $\boldsymbol{\sigma}$  are the Pauli matrices. Note that Eqs. (2) and (3) can be formally derived from the Dirac equation,

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c(\boldsymbol{\alpha} \nabla) \psi + mc^2 \beta \psi, \quad (4)$$

if we suggest that a four component spinor has the form  $\psi_\eta^T = (i\sigma_2 \eta^*, \eta)$  or  $\psi_\xi^T = (\xi, -i\sigma_2 \xi^*)$ , which satisfy the Majorana condition (1). Here we choose the chiral representation of spinors and Dirac matrices:  $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$  and  $\beta = \gamma^0$ . In the following we will use the natural units in which  $\hbar = c = 1$ . In our analysis we just postulate Eqs. (2) and (3).

It should be noticed that the description of Majorana particles in terms of the Weyl spinors is more suitable since the electroweak interaction of elementary particles involves the chiral projections of four component spinors,  $\psi_{L,R}$ , which are equivalent to the Weyl fields  $\eta$  and  $\xi$ . Despite the equal significance of Eqs. (2) and (3), Eq. (2) is used for the description of a massive Majorana neutrino since it was experimentally established that active neutrinos correspond to left-handed fields. That is why we will be mainly interested in Eq. (2). Note that the unitary equivalence of Majorana and Weyl fields was rigorously proved in Ref. [3].

Despite the numerous works devoted to the analysis of Eq. (2) still there is a gap in the understanding of the dynamics of Weyl fields. For the first time a Weyl field was quantized in Ref. [5]. Since then the methods of the quantization of Majorana fermions have not evolved significantly. However in the last few years several works, devoted to the field theory analysis of Majorana particles, appeared. We should mention Ref. [6] where Majorana fermions were studied in connection with the dark matter problem. The quantization of Majorana-like fermions in  $(2+1)$ -dimensional space in a superconductor was made in Ref. [7].

It is known that to carry out a canonical quantization of a field one should construct a Hamiltonian on the basis of an existing Lagrangian [8]. Using the standard variational procedure, Eq. (2) can be formally derived from the Lagrangian [3],

$$\mathcal{L} = i\eta^\dagger (\sigma^\mu \partial_\mu) \eta - \frac{i}{2} m \eta^T \sigma_2 \eta + \frac{i}{2} m \eta^\dagger \sigma_2 \eta^*, \quad (5)$$

where  $\sigma^\mu = (I, -\boldsymbol{\sigma})$  and  $I$  is the unit  $2 \times 2$  matrix.

We can however notice that the mass term in Eq. (5) vanishes if the spinor  $\eta$  is represented as a non-operator field with the commuting  $c$ -number components. Using this fact it was claimed in Ref. [4] that a massive Majorana particle can exist only as an essentially quantum object expressed via anticommuting operators. Therefore the quantization procedure described in Ref. [3] is just the re-expression of already quantized fields in terms of the new variables.

Majorana condition (1) means that the charge conjugated field  $\psi^c$ , corresponding to an antiparticle, is identical to  $\psi$ , which describes a particle. Thus this condition may be considered as the equality of a particle and an antiparticle, which is well defined only in frames of the quantum field theory. This fact was used in Ref. [3] to put forward a vague argument for the necessity of the quantization of a Majorana field. We should however notice that the condition (1) is applied on the four component spinor before its quantization and, as we mentioned above, Eqs. (2) and (3) can be obtained directly from the Dirac equation for a classical spinor field.

Before we proceed a remark should be made on the classical field theory description of a spinor field. The Dirac equation (4) contains the Plank constant  $\hbar$ . Therefore, besides the case of massless fermions, the Dirac equation

always corresponds to a quantum particle. However one can treat the wave function  $\psi$  as a  $c$ -number object and describe its dynamics in frames of the classical field theory. One may speak about a quantized fermion field when  $\psi$  is expressed in terms of the creation and annihilation operators. This terminology is borrowed from Ref. [9].

In the present Letter we develop a consistent theory of a quantum massive Weyl field. We will try to overcome the aforementioned difficulties in the treatment of Majorana particles. First we start with the development of the classical field theory approach for the description of a Weyl field. We propose the classical Hamiltonian for a two component Weyl spinor. Using a variational procedure we obtain the main Eq. (2), which governs the Weyl field evolution. Then we carry out the canonical quantization of a Weil field. We find the plane wave solutions of the wave equations for Weyl fields and calculate their energy using the Hamiltonian proposed. The requirement of the positive definiteness of the energy results in the establishment of the anticommutation expressions for the field amplitudes which turn out to be operators now. Finally we discuss our results.

As we mentioned above the most prominent candidates to be described in terms of Majorana fields are neutrinos. It was experimentally proven that neutrinos are mixed particles whereas the present work is devoted to the description of a single free Weyl field. Nevertheless the results of our work can be easily generalized to include several neutrino generations. Note that the evolution of mixed massive Dirac and Majorana neutrinos was studied in frames of the relativistic quantum mechanics (or classical field theory) to phenomenologically describe neutrino oscillations in vacuum and various external fields (see Ref. [10] and references therein).

To start with the development of the classical field theory approach for the description of a Weyl field we discuss a particular case when  $\eta$  in Eq. (2) does not depend on spatial coordinates,  $\nabla\eta = 0$ . Physically it correspond to a particle at rest. Then we obtain a pair of coupled equations,

$$\dot{\eta} + m\sigma_2\eta^* = 0, \quad \dot{\eta}^* - m\sigma_2\eta = 0, \quad (6)$$

where a dot means the total time derivative.

It was shown in Ref. [11] that, using a standard variational procedure, Eq. (6) can be derived from a Lagrangian with a kinetic term involving a symplectic two-form. In our case the matrix of the two-form  $\sim \sigma_2$  is non-degenerate and we can immediately construct the Lagrangian,

$$L = \frac{1}{2}\dot{\eta}^T\sigma_2\eta - \frac{1}{2}(\dot{\eta}^*)^T\sigma_2\eta^* - m(\eta^*)^T\eta. \quad (7)$$

Note that in the Euler-Lagrange equations the spinors  $\eta$  and  $\eta^*$  should be treated as independent variables. On the basis of the Lagrangian (7) and using the standard technique we find a Hamiltonian for the system,

$$H = m [(\eta^*)^T\sigma_2\pi + (\pi^*)^T\sigma_2\eta], \quad (8)$$

which is written in the symmetric manner with respect to the “coordinates” ( $\eta$  and  $\eta^*$ ) and the momenta ( $\pi$  and  $\pi^*$ ).

Now we restore the coordinate dependence of the spinors. The Hamiltonian (8) becomes a functional expressed as

$$H[\eta, \eta^*, \pi, \pi^*] = \int d^3\mathbf{r} \{ \pi^T(\boldsymbol{\sigma}\nabla)\eta - (\eta^*)^T(\boldsymbol{\sigma}\nabla)\pi^* + m [(\eta^*)^T\sigma_2\pi + (\pi^*)^T\sigma_2\eta] \}. \quad (9)$$

Using the classical field theory version of the canonical Hamilton equations,

$$\dot{\eta} = \frac{\delta H}{\delta \pi} = (\boldsymbol{\sigma}\nabla)\eta - m\sigma_2\eta^*, \quad \dot{\eta}^* = \frac{\delta H}{\delta \pi^*} = (\boldsymbol{\sigma}^*\nabla)\eta^* + m\sigma_2\eta, \quad (10)$$

we obtain Eq. (2) for a massive Weyl field. With help of the second pair of the canonical equations,

$$\dot{\pi} = -\frac{\delta H}{\delta \eta} = (\boldsymbol{\sigma}^*\nabla)\pi + m\sigma_2\pi^*, \quad \dot{\pi}^* = -\frac{\delta H}{\delta \eta^*} = (\boldsymbol{\sigma}\nabla)\pi^* - m\sigma_2\pi, \quad (11)$$

one gets the equations for the canonical momenta. If we introduce the new variables  $\xi = i\sigma_2\pi$  and  $\xi^* = i\sigma_2\pi^*$ , Eq. (11) becomes equivalent to Eq. (3) for  $\xi$ .

Note that Eq. (10) for the “coordinates” does not contain the momenta and vice versa. Thus two groups of variables ( $\eta, \eta^*$ ) and ( $\pi, \pi^*$ ) evolve in time independently. It means that one cannot find the relation between the canonical momenta and the “velocities”,  $\dot{\eta}$  and  $\dot{\eta}^*$ . Moreover if one tries to reconstruct a Lagrangian for this system using a standard recipe,

$$L = \int d^3\mathbf{r} [\pi^T\dot{\eta} + (\pi^*)^T\dot{\eta}^*] - H, \quad (12)$$

then, accounting for Eq. (10), we get that the Lagrangian for our system is trivial:  $L = 0$ . This fact will be discussed later in details.

Using the results of Ref. [3] we find the plane waves solutions of Eqs. (2) and (3) in the following form:

$$\begin{aligned}\eta(x) &= \frac{1}{2} \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2}} \sqrt{1 + \frac{E}{|\mathbf{p}|}} \left[ \left( a_- w_- - \frac{m}{E + |\mathbf{p}|} a_+ w_+ \right) e^{-ipx} \right. \\ &\quad \left. + \left( a_+^* w_- + \frac{m}{E + |\mathbf{p}|} a_-^* w_+ \right) e^{ipx} \right], \\ \xi(x) &= \frac{i}{2} \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2}} \sqrt{1 + \frac{E}{|\mathbf{p}|}} \left[ \left( b_+ w_+ + \frac{m}{E + |\mathbf{p}|} b_- w_- \right) e^{-ipx} \right. \\ &\quad \left. + \left( b_-^* w_+ - \frac{m}{E + |\mathbf{p}|} b_+^* w_- \right) e^{ipx} \right],\end{aligned}\tag{13}$$

where  $p^\mu = (E, \mathbf{p})$ ,  $E = \sqrt{\mathbf{p}^2 + m^2}$  is the energy of a particle, and  $w_\sigma$ ,  $\sigma = \pm$ , are the helicity amplitudes. Here we list some of their useful properties,

$$(\sigma \mathbf{p}) w_\sigma = \sigma |\mathbf{p}| w_\sigma, \quad i\sigma_2 w_\sigma^* = -\sigma w_{-\sigma}, \quad w_\sigma(-\mathbf{p}) = i w_{-\sigma}(\mathbf{p}).\tag{14}$$

The explicit form of the helicity amplitudes can be found in Ref. [12].

In the classical field theory (see, e.g., Ref. [10]) the expansion coefficients  $a_\pm(\mathbf{p})$  and  $b_\pm(\mathbf{p})$  were supposed to be  $c$ -number functions. However now we assume that these objects are commuting or anticommuting operators. The type of statistics will be chosen to provide the positive definiteness of the energy. It should be also noted that we take the different kinds of operators in the decomposition of  $\eta$  and  $\pi$  (or  $\xi$ ) since, as we mentioned above, these fields evolve independently.

On the basis of Eqs. (9), (13), and (14), after a bit lengthy but straightforward calculations we get the Hamiltonian expressed in terms of the operators  $a_\pm(\mathbf{p})$  and  $b_\pm(\mathbf{p})$  and their conjugate,

$$\begin{aligned}H &= \frac{1}{4} \int d^3 \mathbf{p} E \left\{ \left( 1 + \frac{E}{|\mathbf{p}|} \right) \left\{ a_-^*(\mathbf{p}) b_-(\mathbf{p}) + b_-^*(\mathbf{p}) a_-(\mathbf{p}) - a_+(\mathbf{p}) b_+^*(\mathbf{p}) - b_+(\mathbf{p}) a_+^*(\mathbf{p}) \right. \right. \\ &\quad \left. \left. + \left( \frac{m}{E + |\mathbf{p}|} \right)^2 [a_-(\mathbf{p}) b_-^*(\mathbf{p}) + b_-(\mathbf{p}) a_-^*(\mathbf{p}) - a_+^*(\mathbf{p}) b_+(\mathbf{p}) - b_+^*(\mathbf{p}) a_+(\mathbf{p})] \right\} \right. \\ &\quad \left. + i \frac{m}{|\mathbf{p}|} \left\{ e^{-2iEt} [a_-(\mathbf{p}) b_-(-\mathbf{p}) + b_-(-\mathbf{p}) a_-(\mathbf{p}) + b_+(-\mathbf{p}) a_+(\mathbf{p}) + a_+(\mathbf{p}) b_+(-\mathbf{p})] \right. \right. \\ &\quad \left. \left. + e^{2iEt} [a_-^*(\mathbf{p}) b_-^*(-\mathbf{p}) + b_-^*(-\mathbf{p}) a_-^*(\mathbf{p}) + b_+^*(-\mathbf{p}) a_+^*(\mathbf{p}) + a_+^*(\mathbf{p}) b_+^*(-\mathbf{p})] \right\} \right\}.\end{aligned}\tag{15}$$

Now we establish the following relation between the independent operators  $a_\pm(\mathbf{p})$  and  $b_\pm(\mathbf{p})$ :

$$a_\pm(\mathbf{p}) = b_\pm(\mathbf{p}),\tag{16}$$

and the analogous expression for the conjugate operators. We will choose the operators  $a_\pm(\mathbf{p})$  as the basic ones and assume that they obey the anticommutation properties,

$$\{a_\sigma(\mathbf{k}); a_\sigma^*(\mathbf{p})\}_+ = \delta(\mathbf{k} - \mathbf{p}),\tag{17}$$

with all the other anticommutators being equal to zero. In this case the time dependent terms in Eq. (15) are washed out. Using Eqs. (16) and (17) we can recast Eq. (15) into the form

$$H = \int d^3 \mathbf{p} E (a_-^* a_- + a_+^* a_+) + \text{divergent terms},\tag{18}$$

which shows that the total energy of a Weyl field is a sum of the energies of elementary oscillators corresponding to the negative and the positive helicity states.

In the canonical formalism the total momentum of our system can be calculated using the expression,

$$\mathbf{P} = \int d^3 \mathbf{r} [(\eta^*)^T \nabla \pi^* - \pi^T \nabla \eta],\tag{19}$$

which is obtained by the spatial integration of the  $T^{i0}$  component of the energy-momentum tensor  $T^{\mu\nu}$ . Omitting the detailed calculations and with help of Eqs. (13), (14), (16), and (17) we get the following formula for the quantized momentum of the Weyl field:

$$\mathbf{P} = \int d^3\mathbf{p} \mathbf{p} (a_-^* a_- + a_+^* a_+) + \text{divergent terms}, \quad (20)$$

which has the analogous structure as Eq. (18).

There is, however, another way to quantize a Weyl field. Instead of Eq. (16) we may choose the following relation between the operators:

$$a_{\pm}(\mathbf{p}) = b_{\mp}(\mathbf{p}), \quad (21)$$

with the condition (17) still being held true for the operators  $a_{\pm}$ . In this case the time dependent terms in Eq. (15) are also equal to zero. To diagonalize the remaining time independent term in Eq. (15) we introduce the new operators  $c_{\pm}$  by means of the Bogoliubov transformation,

$$a_- = \frac{1}{\sqrt{2}}(c_- - c_+^*), \quad a_+ = \frac{1}{\sqrt{2}}(c_- + c_+^*). \quad (22)$$

One can check by a direct calculation that the new operators also satisfy the canonical anticommutation rules. Finally the quantized Hamiltonian is expressed as

$$H = \int d^3\mathbf{p} E(c_-^* c_- + c_+^* c_+) + \text{divergent terms}. \quad (23)$$

One can also show that, after the quantization in terms of the operators  $c_{\pm}$ , the total momentum takes the form,

$$\mathbf{P} = \int d^3\mathbf{p} \mathbf{p} (c_-^* c_- + c_+^* c_+) + \text{divergent terms}. \quad (24)$$

We can see that Eqs. (23) and (24) have the same structure as Eqs. (18) and (20) respectively.

In summary we mention that in the present Letter we have carried out a consistent canonical quantization of a massive Weyl field. Two major results have been obtained. First we have constructed a classical field theory approach for the description of the Weyl field dynamics. The classical field theory was applied in the form of a canonical Hamilton formalism. We have derived a classical Hamiltonian (9) and using a standard variational procedure have obtained the main Eqs. (2) and (3) for Weyl spinors. Note that previously Weyl fields were considered as essentially quantum objects [3]. In Ref. [13] we have found classical solutions of Eq. (2) to describe Majorana neutrino oscillations in vacuum. Now the classical field theory approach for the treatment of Weyl spinors is fully substantiated. Although we do not doubt that our world is quantum, numerous processes may be also described within the classical physics (see Ref. [14] for many interesting examples).

We have mentioned that it is impossible to construct a classical Lagrangian for the system of Weyl spinors whereas the classical Hamiltonian is well defined (9). It is known that the Lagrange and the Hamilton formalisms for the treatment of classical particles are almost equivalent: for the existing Lagrangian one can always construct a Hamiltonian, but the opposite statement is not true. The example of the real physical process, the rays of light propagation in a medium, which can be described only within the canonical formalism since the Lagrangian for such a system is trivial,  $L = 0$ , is given in Ref. [15]. Thus Weyl fields dynamics can be attributed to a class of physical systems which is treated only using the Hamilton formalism. Hamiltonians analogous to Eq. (9), resulting in the first order evolution equations, were discussed in Ref. [16] for the studies of nonlinear waves in frames of the Hamilton formalism.

The second important result obtained in the present Letter is the new interpretation of the Weyl fields quantization. We have mentioned that evolution equations for “coordinates” (10) and canonical momenta (11) turned out to be disentangled. Thus in frames of the Hamilton formalism one has an additional independent degree of freedom. While quantizing the system, we have established the link between the degrees of freedom to provide the correct form of the total energy (18) or (23) and the momentum (20) or (24) by imposing the conditions (16) or (21). We mentioned that the wave equations for the canonical momenta (11) are equivalent to those for the right-handed Weyl spinor  $\xi$  [see Eq. (3)]. Therefore Eqs. (16) and (21) can be considered as the quantum condition of the identity of particles (left-handed spinors  $\eta$ ) and antiparticles (right-handed spinors  $\xi$ ) since for Majorana spinors the following relation is valid:  $(\psi_L)^c = \psi_R$ .

It is interesting to notice that one can quantize a Weyl field by two independent ways: using the operators  $a_{\pm}$  [see Eqs. (16)-(20)] or using the operators  $c_{\pm}$  [see Eqs. (21)-(24)]. The operators  $a_{\pm}$  in the first quantization scheme or

the operators  $c_{\pm}$  correspond to the physical degrees of freedom measured by an observer since the energy and the momentum are expressed via these operators. Thus the physical degrees of freedom are the superposition particles and antiparticles states. Thus Eq. (16) or Eqs. (21) and (22) may be considered as the quantum analog of the Majorana condition (1).

A remark on the weight coefficient in Eq. (13) should be made. One notices that the function  $\rho(\mathbf{p}) = \sqrt{1 + E/|\mathbf{p}|}$  becomes singular at  $|\mathbf{p}| \rightarrow 0$ . We should however notice that  $\rho$  is a square-integrable function. Indeed, for  $m \neq 0$  we get

$$4\pi \int_0^{\infty} \mathbf{p}^2 d|\mathbf{p}| \rho^2(\mathbf{p}) < \infty. \quad (25)$$

Thus the Fourier transform in Eq. (13) is well defined. We also mention that in solving of linear problems, e.g., related to oscillations of Majorana neutrinos (see Ref. [10]), one may choose a non-singular weight coefficient like in Ref. [3].

## ACKNOWLEDGMENTS

I am very thankful to S. Forte for bringing the key Ref. [11] to my attention and to J. Maalampi for stimulating discussions.

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- [1] R. N. Mohapatra and A. Yu. Smirnov, *Ann. Rev. Nucl. Part. Sci.* **56**, 569 (2006), hep-ph/0603118.
  - [2] E. Andreotti, *et al.* (CUORICINO Collaboration), *Astropart. Phys.* **34**, 822 (2011), arXiv:1012.3266 [nucl-ex]; J. Argyriades, *et al.* (NEMO Collaboration), *Phys. Rev. C* **80**, 032501 (2009), arXiv:0810.0248 [hep-ex]; T. Bloxham, *et al.* (COBRA Collaboration), *Phys. Rev. C* **76**, 025501 (2007), arXiv:0707.2756 [nucl-ex].
  - [3] M. Fukugita and T. Yanagida, *Physics of neutrinos and applications to astrophysics* (Berlin, Springer, 2003), pp. 289–319.
  - [4] J. Schechter and J. W. F. Valle, *Phys. Rev. D* **24**, 1883 (1981).
  - [5] K. M. Case, *Phys. Rev.* **107**, 307 (1957).
  - [6] D. V. Ahluwalia and S. P. Horvath, *JHEP* **1011**, 078 (2010), arXiv:1008.0436 [hep-ph]; D. V. Ahluwalia, C.-Y. Lee, and D. Schrott, *Phys. Rev. D* **83**, 065017 (2011), arXiv:0911.2947 [hep-ph].
  - [7] C. Chamon, *et al.*, *Phys. Rev. B* **81**, 224515 (2010), arXiv:1001.2760 [cond-mat.str-el].
  - [8] S. Weinberg, *The quantum theory of fields: foundations* (Cambridge, Cambridge Univ. Press, 1996), 2nd ed., pp. 292–338.
  - [9] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the theory of quantized fields* (NY, Wiley, 1980), 3rd ed., pp. 10–89.
  - [10] M. Dvornikov, to be published in *Neutrinos: properties, sources and detection*, ed. by J. P. Greene, (NY, NOVA Science Publishers, 2011), arXiv:1011.4300 [hep-ph].
  - [11] L. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).
  - [12] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum electrodynamics* (Oxford, Pergamon, 1980), 2nd ed., p. 86.
  - [13] M. Dvornikov and J. Maalampi, *Phys. Rev. D* **79**, 113015 (2009), arXiv:0809.0963 [hep-ph].
  - [14] D. Giulini, *et al.*, *Decoherence and the appearance of a classical world in quantum theory* (Berlin, Springer, 2003), 2nd ed.
  - [15] L. D. Landau and E. M. Lifshitz, *The classical theory of fields* (Amsterdam, Butterworth-Heinemann, 1994), 4th ed., pp. 140–143.
  - [16] V. E. Zakharov and E. A. Kuznetsov, *Phys. Usp.* **40**, 1087 (1997).